# Static Aeroelasticity and the Flying Wing, Revisited

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SINCE the appearance of their original paper,[1] the authors have received several very cogent questions pertaining to the practical importance and physical significance of the theoretical aeroelastic "instabilities" discussed in that paper. In addition to highlighting the aeroelastic aspects of tailless-aircraft design, Ref. 1 intended to describe, in a simplified fashion, some phenomena which may be peculiar either to flying wings or to oblique winged aircraft.[2] These phenomena included the role of aileron controls in aeroelastic behavior and the influence of asymmetrical sweep on the aeroelastic behavior of oblique wings. The purpose of the present Note is to summarize an approach which the authors believe to be helpful in dealing with the issues which have been raised.

It is useful to recall Ziegler's discovery[3] that stability boundaries for nonconservative linear systems (e.g., the cantilever beam with a follower force load at the free end) are best determined from a study of the *time dependent behavior* of the free motion of the system. This point is exemplified by a reanalysis of the torsional effects on a uniform property straight wing, rolling about a pin as shown in Figs. 5 and 6 of Ref. 1. Notation will follow that used in the antecedent paper.

Consider initially the roll dynamics of the wing with undeflected aileron, but assume that oscillation periods are sufficiently long to permit the quasi-static treatment of wing torsional deformation; in this case, the $\ddot{\theta}$ term is negligible. If the line of centers of mass coincides with the elastic axis, the torsional deformation equation may be expressed (in nondimensional form) as

$$\frac{\partial^2\theta}{\partial\eta^2} + k^2\left[\theta + \frac{pl}{V}\eta\right] = 0 \tag{1}$$

The partial derivative reflects the fact that $\Theta = \Theta(\eta,\tau)$, where $\tau$ is a nondimensional time variable

$$\tau = \frac{Vt}{l} \tag{2}$$

In preference to a static constraint of zero rolling moment, let Eq. (1) be solved simultaneously with the roll dynamics equation. After nondimensionalization and the inclusion of the twist contribution to the rolling moment, the equation for roll moment dynamic equilibrium reads

$$\frac{d}{d\tau}\left(\frac{pl}{V}\right) + \frac{a_o}{3i_x}\left(\frac{pl}{V}\right) = -\frac{a_o}{i_x}\int_0^1 \theta(\eta,\tau)\eta d\eta \tag{3}$$

Here $i_x = I_x/\rho_\infty cl^4$ is a dimensionless form of $I_x$, the mass moment of inertia of the wing about the pin.

With cantilever boundary conditions on twist and an initial rolling velocity

$$p_o = p(0) \tag{4}$$

Eqs. (1–3) are easily seen to yield a first-order system in time. The solution

$$p(\tau) = p_o\left\{\exp\left[-\frac{a_o}{k^3 i_x}(\tan k - k)\tau\right]\right\} \tag{5}$$

is useful for interpreting the significance (or insignificance) of the roots of the characteristic equation, Eq. (12), in Ref. 1. To this end, Fig. 1 plots the dynamic pressure parameter $2k/\pi$ vs the "time constant" $\tau_o$, where

$$\tau_o = -k^3 i_x/a_o(\tan k - k)$$

is associated with roll response. From this presentation it immediately becomes clear that any instability due to an eigenvalue of the system described by Eqs. (1–3) at $\tan k = k$ is masked by what happens at the much lower dynamic pressure eigenvalue $k = \pi/2$. Ordinary cantilever torsional divergence is the aeroelastic threat to this wing.

Next, let us consider what happens if the full span ailerons are applied in a wing-leveling mode. For instance, one might select the elementary control law

$$\delta_o = K\phi \tag{6}$$

where

$$\phi = \int_0^\tau \frac{pl}{V} d\tau \tag{7}$$

is the bank angle and $K(K > 0)$ is a gain constant. (It must be remarked that no essential change in aeroelastic behavior occurs if Eq. 6 is replaced by some realistic dynamic control law, so long as the proportionality of $\delta_o$ to $\phi$ is preserved as the frequency goes to zero.)

Under these circumstances, the torsional deformation differential equation is merely modified by some aileron terms like those in Eq. (13) of Ref. 1. The solution to this equation has the form of Eq. (19) of Ref. 1, with the term $-pl\eta/V$ appended. When this result and the control law given in Eq. (6) are added to the roll dynamic equilibrium equation, the result is

$$\frac{d}{d\tau}\left(\frac{pl}{V}\right) + \frac{a_0}{3i_x}\left(\frac{pl}{V}\right) = -\frac{c_{l\delta}}{2i_x}\delta_o - \frac{a_o}{i_x}\int_0^1 \theta(\eta,\tau)\eta d\eta \tag{8}$$

The bank angle is then found to obey the following second-order differential equation

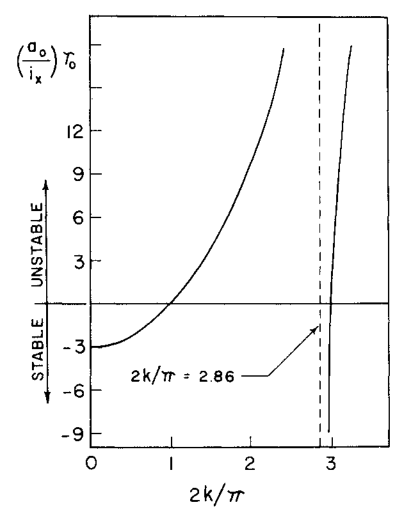


**Fig. 1 Variation of the time constant $\tau_o$ with dynamic pressure parameter $2k/\pi$.**

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$$\frac{d^2\phi}{d\tau^2} + \frac{a_o}{k^3 i_x}[\tan k - k]\frac{d\phi}{d\tau} + \frac{K}{i_x}\left\{[c_{l\delta} + \frac{c}{e}c_{M\delta}]\left[\frac{1-\cos k}{k^2 \cos k}\right] - \frac{c c_{M\delta}}{2e}\right\}\phi = 0 \qquad (9)$$

The transient solutions of Eq. (9) behave in a manner that can be inferred from the variation with k of the dimensionless "damping coefficient" and "spring constant." By analogy with Eq. (5), the damping coefficient undergoes a sign change at $k = \tan k$. However, this is dominated in practice by the singularity due to torsional divergence at $k = \pi/2$. Perhaps more interesting, however, is the changeover from positive to negative "spring constant" as $k$ increases. When the gain $K$ is large enough, this change in sign signals an unstable passage from damped oscillatory to exponentially divergent roll perturbation. By reference to pp. 300–310 of Ref. 4, the stability boundary is found to coincide with the dynamic pressure for aileron reversal, at which point the roll-controllability of the system vanishes. Whenever the group of terms $[c_l + c_M\, c/e]$ is negative (corresponding to $f < 0$ in Ref. 1) this phenomenon appears below the torsional divergence $q$.

For a representative set of system parameters, with $K$ chosen large enough to make the rigid wing roll eigenvalues oscillatory, Fig. 2 displays how the dimensionless natural frequency and critical damping ratio from Eq. (9) vary with $k$.

Turning to the case of an oblique wing free to roll about an axis parallel to the airstream, we may treat the dynamic behavior in a manner similar to that just described. As in Ref. 1, the aeroelastic analysis is simplified by the assumption that the elastic axis and line of aerodynamic centers coincide; in addition, the wing is uncambered. With these assumptions, only bending flexibility is important. Because the qualitative behavior of the configuration is to be examined, strip theory aerodynamics again will be used, despite some well-known shortcomings of the theory when used for swept wing airload prediction or for prediction of loading caused by aileron deflection.

If chordwise cross-sections are examined, the nondimensional governing differential equation for the bending deformation of a constant property wing with full-span ailerons, rolling on a pin, is found to be

$$\frac{\partial^4 w}{\partial \eta^4} + (\lambda \tan\Lambda)\frac{\partial w}{\partial \eta} - \lambda\left(\frac{c_{l\delta}}{c_{l\alpha}}\right)\delta(\eta) - \lambda\left(\frac{pl}{V}\right)\eta - \dot{p}\left(\frac{ml^4}{EI}\cos\Lambda\right)\eta = 0 \qquad (10)$$

where $\lambda = qcc_{l\alpha}\, l^3\cos^2\Lambda/EI$, $w(\eta,t)$ is the elastic deflection of the wing and $m(\eta)$ is the wing mass per unit length. Once again, the oscillation periods are assumed sufficiently large so that the $\ddot{w}$ term can be ignored. We cannot ignore the constant mass distribution $m(\eta)$ without introducing a significant error, as will be shown later.

Equation (10) is to be solved simultaneously with the oblique wing roll dynamics equation. This latter equation reads

$$\frac{dp}{dt} + \frac{2}{3}\gamma\frac{pl}{V} = -\gamma\left(\frac{c_{l\delta}}{c_{l\alpha}}\right)\delta_o + \gamma\tan\Lambda\int_{-1}^{1}\frac{\partial w}{\partial\eta}\eta\, d\eta \qquad (11)$$

where $\gamma = qcc_{l\alpha}\, l^2\cos^3\Lambda/I_x$; $I_x$ is the mass moment of inertia of the wing about the roll axis; and the ailerons are deflected asymmetrically so that

$$\delta(\eta) = \begin{cases} \delta_o & 0 < \eta \le 1 \\ -\delta_o & -1 \le \eta < 0 \end{cases}$$

To make the problem more realistic, it is further assumed that part of the moment of inertia is concentrated on the roll axis, as would be the case if a fuselage were present. With this assumption, $I_x = I_f + I_w$, where $I_f$ is the roll mass moment of inertia of the fuselage while $I_w$ is the wing roll mass moment of inertia; $I_w$ is found, by simple integration, to be

$$I_w = (2/3)ml^3\cos^2\Lambda \qquad (12)$$

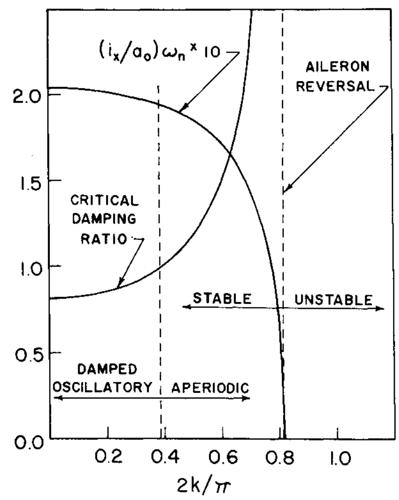


**Fig. 2 Typical variation of undamped dimensionless natural frequency and critical damping ratio with dynamic pressure parameter $2k/\pi$; $f = -1.0$, $K = a_o^2/12c_{l\delta}i_x$.**

The exact solution to Eq. (10) may be found if the wing elastic deformation derivative $\partial w/\partial\eta$ is assumed to be zero at the roll axis attachment point. This solution has a form similar to that given in Ref. 4, pp. 312–314. For the sake of brevity, the analytic solution and the related integrand in Eq. (11) are not reproduced here.

If, as in Eq. (6), $\delta_o$ is assumed proportional to the bank angle $\phi$, Eq. (11) may be written as

$$\frac{d^2\phi}{dt^2} + \left(\frac{2}{3}\frac{\gamma l}{V}D\right)\frac{d\phi}{dt} + \left(\frac{c_{l\delta}}{c_{l\alpha}}\right)FK\phi = 0 \qquad (13)$$

The symbols $D$ and $F$ represent terms which arise from the combination of like terms in Eq. (11) after the required integration has been performed. Both $D$ and $F$ are exponential functions of the aeroelastic parameter $\lambda \tan\Lambda$. In addition, $D$ is also a function of the ratio $I_w/I_x$. The behavior of the damping factor $D$ is shown in Fig. 3, where $I_w/I_x$ has the value 0.50.

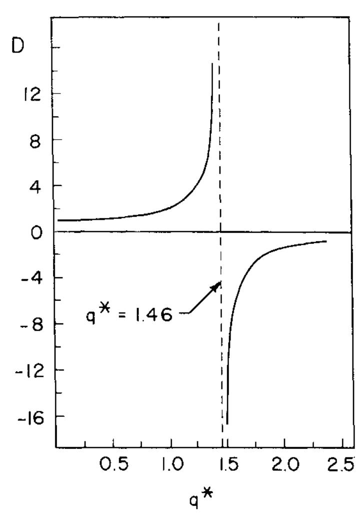


**Fig. 3 Variation of the damping factor $D$ with the nondimensional dynamic pressure parameter $q^*$; $q^* = q/q_{DIV}$; $q_{DIV} = 6.33EI/cc_{l\alpha}l^3 \sin\Lambda \cos\Lambda$.**

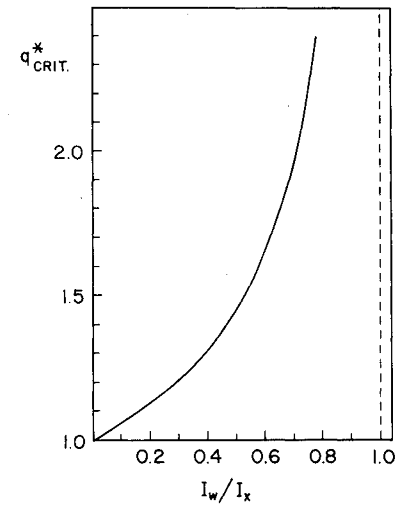


**Fig. 4 The effect of the mass moment of inertia ratio $I_w/I_x$ on the roll instability boundary for the oblique wing.**

In Fig. 3, $D$ is plotted against the variable $q^*$. $q^*$ represents the dynamic pressure $q$ divided by the divergence dynamic pressure of a similar clamped sweptforward wing which does not have freedom to roll. If the gain $K$ is large enough, the temporal behavior of the bank angle is that of a damped oscillation until the value $q^* = 1.46$. At that point the damping factor $D$ changes sign (as does $F$), with the result that the roll motion becomes exponentially divergent. This critical value of $q^*$ is greater than unity, signifying that this dynamic instability occurs at a value *above the clamped divergence speed.*

Figure 4 illustrates the effect of the ratio $I_w/I_x$ on the dynamic instability. From this figure two things are readily apparent. If all the mass is concentrated in the fuselage ($I_w = 0$), the dynamic instability occurs at the clamped divergence speed. On the other hand, if all the mass is in the wing ($I_w = I_x$) no dynamic instability can occur in this problem. It should be remarked also that, at a fixed value of $q$, the sweeping of the wing causes $q_{\mathrm{DIV}}$ to decrease (with the resultant increase in $q^*$) and also causes $I_w/I_x$ to decrease.

The variable $F$ is equal to zero at $\lambda \tan\Lambda = 27.45$. This behavior is related to a phenomenon which was referred to as "wing-aileron divergence" in Ref. 1. When $F$ is equal to zero, the solution to Eq. (13) takes on the form

$$\phi(t) = A + Be^{\alpha t} \tag{14}$$

where $A$ and $B$ are arbitrary constants and $\alpha$ is a positive constant. If $B$ is zero, there is no roll velocity and the result corresponds to the case of an oblique wing which uses asymmetrical aileron deflection to maintain a constant bank angle. But, as $q$ (or $\lambda \tan\Lambda$) increases, the ailerons lose their effectiveness and additional aileron input is necessary to preserve static equilibrium. Finally, near $\lambda \tan\Lambda = 27.45$ the ailerons must be deflected to large angles to preserve static equilibrium. These large aileron deflections induce large bending deformation, thus the term "wing-aileron divergence."

It has been pointed out by Nisbet[5] that, in fact, the theoretical value $\lambda \tan\Lambda = 27.45$ must correspond to the aileron control reversal point for the oblique wing. This result was unanticipated by the first author at the time Ref. 1 was published, since an aileron on a clamped sweptforward wing (considering bending deformation only) will not reverse. The correctness of Nisbet's observation is borne out by a conventional analysis of oblique wing aileron effectiveness.

### Conclusions

This Note has examined the roll dynamic behavior of the three example problems considered in Ref. 1. The results obtained in Ref. 1 through the use of static equilibrium analysis are seen to be related to the roll dynamic stability behavior of these configurations. The authors emphasized in Ref. 1 that the static aeroelastic "instabilities" uncovered in that paper "in actuality lead, not to structural failures, but to large amounts of twist and bending." These static "instabilities" are, when seen from a dynamic response viewpoint, seen to be caused by control ineffectiveness.

The idealized dynamic analysis of the oblique wing, free to roll, has led to results which are extremely interesting. The analysis shows that the static divergence instability which occurs for symmetrical or clamped sweptforward wings is modified by roll freedom. The instability found in this highly idealized analysis is still of an aperiodic nature, but at a speed above the conventional divergence speed. The magnitude of this difference is seen to be a function of the roll mass moment of inertia ratio $I_w/I_x$ and may be quite significant.

Although the present analysis stems from a greatly simplified and somewhat different set of assumptions (notably that the ailerons furnish a restoring force), its basic results tend to agree with the results in Ref. 2. In Ref. 2 a flutter (i.e., divergent oscillation), instability slightly above the cantilever divergence speed is shown to exist if the aircraft is given roll freedom. Although the analysis in the present study and in Ref. 2 are different in many respects, they both show a significant upward modification of the speed at which aeroelastic instability occurs if roll freedom is permitted. Further studies are currently underway to determine the effect of the addition of pitch and plunge freedom to a realistic aeroelastic model for which three-dimensional unsteady aerodynamic theory, rather than strip theory, is used. The results of this further analysis should shed additional light on this fascinating problem.

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